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$\sigma_k(F_m) = F_n$

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Abstract. Let $\sigma_k(n)$ be the sum of the kth powers of the divisors of n. Here, we prove that if $(F_n)_{n\geq 1}$ is the Fibonacci sequence, then the only solutions of the equation $\sigma_k(F_m) = F_n$ in positive integers $k \geq 2$, m and n have $k = 2$ and $m \in \{1, 2, 3\}$. The proof uses linear forms in two and three logarithms, lattice basis reduction, and some elementary considerations.

1. Introduction

Classical arithmetic functions of a positive integer n are the Euler function $\phi(n)$, the sum of divisors function $\sigma(n)$, etc. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Equations involving arithmetic functions of Fibonacci numbers were investigated in a number of recent papers. In [**6**], it was shown that there is no perfect Fibonacci number. That is, there is no Fibonacci number F_n such that $\sigma(F_n) = 2F_n$. In [**1**], it was shown that there is no multiperfect Fibonacci number $F_n > 1$; that is, F_n is not a divisor of $\sigma(F_n)$. Similar equations with Fibonacci numbers involving the Euler function $\phi(n)$ instead of the sum of divisors function $\sigma(n)$ were also investigated. For example, in [**7**] it was shown that if $F_n > 1$ and $\phi(F_n)$ divides $F_n - 1$, then F_n is prime, while in [**8**] it was shown that if $\phi(F_n)$ is a Fibonacci number, then $n = 1, 2, 3, 4$.

Here, for a positive integer k we put $\sigma_k(n)$ for the sum of the kth powers of the divisors of n and look at the Diophantine equation

$$\sigma_k(F_m) = F_n. \tag{1}$$

Since $F_1 = F_2 = 1$, equation (1) has the trivial solutions $m,\ n \in \{1, 2\}$ for any $k \geq 1$. We prove the following result.

Theorem 1. *The only nontrivial solution with $k \geq 2$ of equation (1) is $(k, m, n) = (2, 3, 5)$.*

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We could not succeed in finding all the solutions to $\sigma_k(F_m) = F_n$ in the case $k = 1$, although in [**4**] it was shown that the set of such m is of asymptotic density zero. The plan of the proof is to first bound k, then bound m, and then finish off the job. We start with some elementary considerations.

2. Elementary Considerations

We record that the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{holds for all} \quad n \geq 0, \tag{2}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. The companion Lucas sequence $(L_n)_{n\geq 0}$ has $L_0 = 2,\ L_1 = 1$ and obeys the same recurrence relation $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Its Binet formula is

$$L_n = \alpha^n + \beta^n \qquad \text{for all} \quad n \geq 0.$$

There are many relations involving the Fibonacci and Lucas numbers such as $F_{2n} = F_n L_n$ for all $n \geq 0$. We shall freely use them as we will find it convenient to do so.

For a positive integer k we write $z(k)$ for the order of appearance of k in the Fibonacci sequence, which is the minimal positive integer ℓ such that $k \mid F_\ell$. This always exists and has the important property that $k \mid F_m$ if and only if $z(k) \mid m$.

3. The Case of Small m

Assume throughout that (1) holds. We show in an elementary way that except for the solution $(k, m, n) = (2, 3, 5)$ to equation (1) presented in the statement of the theorem, we must have $m \geq 9$.

Suppose first that $m \in \{3, 4, 5, 6, 7\}$. Since $F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13$, it follows that $F_m = p^\gamma$, where $p \in \{2, 3, 5, 13\}$ and $\gamma \in \{1, 3\}$. It is well-known that for such values of p we have that $p \| F_{z(p)}$. Furthermore, $z(p^u) = z(p)p^{u-1}$ holds for all such p and $u \geq 1$, except when $p = 2$ and $u \geq 3$, for which we have $z(2^u) = 3 \times 2^{u-2} = z(2) \times 2^{u-2}$. We now use the known formulas

$$\begin{aligned}
F_{4\ell} - 1 &= F_{2\ell+1} L_{2\ell-1} \\
F_{4\ell+1} - 1 &= F_{2\ell} L_{2\ell+1} \\
F_{4\ell+2} - 1 &= F_{2\ell} L_{2\ell+2} \\
F_{4\ell+3} - 1 &= F_{2\ell+2} L_{2\ell+1}
\end{aligned}$$

valid for all positive integers ℓ to deduce that

$$\sigma_k(F_m) - 1 = F_n - 1 = F_{(n-\delta)/2} L_{(n+\delta)/2} \qquad \text{with some} \qquad \delta \in \{\pm 1, \pm 2\} \tag{3}$$

such that $n \equiv \delta \pmod 2$. The left hand side in (3) above is a multiple of p^k for some $p \in \{2, 3, 5, 13\}$. Hence, because $F_{n\pm\delta} = F_{(n\pm\delta)/2} L_{(n\pm\delta)/2}$, we have $p^k \mid F_{n-\delta} F_{n+\delta}$.

If n is odd, we then have that $p^k \mid F_{n-1}F_{n+1}$ and $\gcd(F_{n-1}, F_{n+1}) = 1$. Thus, either $p^k \mid F_{n-1}$, or $p^k \mid F_{n+1}$. In particular, $n+1 \geq z(p^k)$. Since

$$z(p^k) \in \{3 \times 2^{k-1},\ 3 \times 2^{k-2},\ 4 \times 3^{k-1},\ 5^k,\ 6 \times 13^{k-1}\},$$

it follows that $n \geq 3\times 2^{k-2}-1$. On the other hand, if n is even, then $p^k \mid F_{n-2}F_{n+2}$. The greatest common divisor of $F_{n-2}F_{n+2}$ divides $F_4 = 3$. Thus, if $p \neq 3$, then p^k divides one of F_{n-2} and F_{n+2}, while if $p = 3$, then p^{k-1} divides one of F_{n-2} or F_{n+2}. A similar argument as above shows that $n+2 \geq 3 \times 2^{k-2}$. Hence, $n \geq 3 \times 2^{k-2} - 2$. It is easy to see that for our cases

$$\sigma_k(F_m) \leq 1 + 13^k < 13^k\alpha \qquad \text{holds for all} \quad m \in \{3,4,5,6,7\}.$$

Using the fact that the inequality $F_n \geq \alpha^{n-2}$ holds for all positive integers n, we get

$$13^k\alpha > \sigma_k(F_m) = F_n > \alpha^{n-2} \geq \alpha^{3\times 2^{k-2}-4},$$

therefore

$$k\left(\frac{\log 13}{\log \alpha}\right) > 3 \times 2^{k-2} - 5,$$

implying that $k \leq 5$. A quick check confirms that there is no other solution (k, m, n) to equation (1) in this range than the one mentioned in the conclusion of the theorem.

There is a similar elementary way to handle the case $m = 8$ also. Namely, for $m = 8$ we have

$$F_n = \sigma_k(F_8) = \sigma_k(21) = (1+3^k)(1+7^k). \tag{4}$$

If k is even, then the right hand side above is congruent to 4 modulo 8, which is impossible because no Fibonacci number is congruent to 4 modulo 8. If k is odd, then $32 = 4 \times 8 = (1+3) \times (1+7) \mid (1+3^k)(1+7^k)$, so that $32 \mid F_n$, therefore $z(32) = 24$ divides n. Since $3 \mid F_{24}$, we get that $3 \mid (1+3^k)(1+7^k)$, which is impossible because the number $(1+3^k)(1+7^k)$ is congruent to 2 modulo 3. Hence, there are no solutions (k, m, n) to equation (1) with $m = 8$ either.

Such elementary arguments can be applied for other fixed small values of m (such as $m = 11$ for which $F_{11} = 89$ is prime), but they already fail for $m = 9$. Throughout the next sections, we use more sophisticated methods to deal with large values of k and m.

4. A Linear Form in Logarithms

Here, we explain how to use a linear form in logarithms to get an inequality involving m and k. This puts a bound on k in terms of m. We shall use this together with lattice basis reduction to deal with the cases when $m \leq 130$. Start with the inequality

$$n^k \leq \sigma_k(n) = n^k \sum_{d\mid n} \frac{1}{d^k} < n^k\zeta(k). \tag{5}$$

Observe that since $k \geq 2$, we have

$$\begin{aligned}
\zeta(k) &= 1 + \frac{1}{2^k} + \sum_{m\geq 3} \frac{1}{m^k} < 1 + \frac{1}{2^k} + \int_2^{\infty} \frac{dt}{t^k} \\
&= 1 + \frac{1}{2^k} + \left(-\frac{1}{(k-1)t^{k-1}} \Big|_{t=2}^{t=\infty} \right) \\
&= 1 + \frac{1}{2^k} + \frac{1}{(k-1)2^{k-1}} \leq 1 + \frac{3}{2^k}. \qquad (6)
\end{aligned}$$

Combining estimates (5) (with n replaced by F_m) and (6), we get that

$$\left| F_n - F_m^k \right| < \frac{3F_m^k}{2^k}.$$

Using also formula (2) together with the fact that $\alpha - \beta = \sqrt{5}$ and $\beta = -\alpha^{-1}$, we get

$$\left| \frac{\alpha^n}{\sqrt{5}} - F_m^k \right| < \frac{1}{\sqrt{5}\alpha^n} + \frac{3F_m^k}{2^k}, \qquad (7)$$

or

$$\left| \alpha^n 5^{-1/2} F_m^{-k} - 1 \right| < \frac{1}{\sqrt{5}\alpha^n F_m^k} + \frac{3}{2^k}.$$

Observe that since $m \geq 9$, we get that $\sqrt{5}F_m^k > F_9^k > 2^k$. Hence, we have that

$$\left| \alpha^n 5^{-1/2} F_m^{-k} - 1 \right| < \frac{1}{2^k} + \frac{3}{2^k} = \frac{1}{2^{k-2}}. \qquad (8)$$

Recall that for an algebraic number η having

$$f(X) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)})$$

as minimal polynomial over the integers, its logarithmic height is defined as

$$h(\eta) = \frac{1}{d} \left(\log |a_0| + \sum_{i=1}^{d} \log \left(\max \left\{ \left| \eta^{(i)} \right|, 1 \right\} \right) \right).$$

With this notation, Matveev [**9**] proved the following deep theorem:

Theorem 2. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\eta_1, \ldots, \eta_\ell$ be nonzero elements of $\mathbb{K}$, and $b_1, \ldots, b_\ell$ rational integers. Put*

$$B = \max\{|b_1|, \ldots, |b_\ell|\}$$

and

$$\Lambda = 1 - \prod_{i=1}^{\ell} \eta_i^{b_i}.$$

Let $A_1, \ldots, A_\ell$ be real numbers such that

$$A_j \geq \max\{Dh(\eta_j), |\log \eta_j|, 0.16\}, \qquad j = 1, \ldots, \ell.$$

Then, assuming that $\Lambda \neq 0$, we have

$$\log |\Lambda| > -3 \cdot 30^{\ell+4} (\ell+1)^{5.5} D^2 (1 + \log D)(1 + \log(\ell B)) \prod_{j=1}^{\ell} A_j.$$

We apply Matveev's Theorem to the expression

$$\Lambda = \alpha^n 5^{-1/2} F_m^{-k} - 1.$$

This is not zero, since otherwise $\alpha^{2n} = 5F_m^{2k}$ would be a integer, which is impossible for $n > 0$. We can take $\ell = 3$, $\eta_1 = \alpha$, $\eta_2 = \sqrt{5}$, $\eta_3 = F_m$. We can also take $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ which has degree $D = 2$ over $\mathbb{Q}$. Since $h(\eta_1) = \log(\alpha)/2$ and $h(\eta_2) = (\log 5)/2$, it follows that we can take $A_1 = \log\alpha$, $A_2 = \log 5$ and $A_3 = 2\log F_m < 2m\log\alpha$. Observe further that since

$$\alpha^{n-1} > F_n = \sigma_k(F_m) > F_m^k > (\alpha^{m-2})^k = \alpha^{(m-2)k},$$

we get that $n - 1 > (m-2)k$. Thus,

$$n \geq (m-2)k, \tag{9}$$

and since $m \geq 9$, we get that $B = \max\{1, k, n\} = n$. Note further that

$$\alpha^{n-2} < F_n = \sigma_k(F_m) < F_m^k \left(1 + \frac{3}{2^k}\right) < 2F_m^k < \alpha^{2+k(m-1)},$$

so that $n < 4 + km - k$, therefore $n \leq mk + 1$. Hence, we get, by Theorem 2 that

$$\log|\Lambda| > -3 \times 30^7 \times 4^{5.5} \times 4(1 + \log 2)(1 + \log(3mk + 3))(\log\alpha)(\log 5)(2m\log\alpha).$$

On the other hand, by inequality (8), we have that

$$\log|\Lambda| < -(k-2)\log 2.$$

Thus, we get that

$$\begin{aligned} k - 2 &< 3 \times 30^7 \times 2^{14}(1 + \log 2)(\log 5)(\log\alpha)^2(\log 2)^{-1} m(1 + \log(3mk + 3)) \\ &< 9.8 \times 10^{14} m(1 + \log(3mk + 3)). \end{aligned} \tag{10}$$

Observe that since $m \geq 9$, $k \geq 2$, we have

$$\begin{aligned} 1 + \log(3mk + 3) &= 1 + \log 3 + \log(mk) + \log\left(1 + \frac{1}{mk}\right) \\ &\leq 1 + \log 3 + \log\left(1 + \frac{1}{18}\right) + \log(mk) \\ &< 2.3 + \log(mk). \end{aligned}$$

Since $\log(mk) \geq \log 18 > 2.3$, we get that

$$1 + \log(3mk + 3) < 2.3 + \log(mk) < 2\log(mk).$$

Hence,

$$k < 9.8 \times 10^{14} \times 2m\log(mk) + 2 < 2 \times 10^{15} m\log(mk).$$

Let us see an easy consequence. If $k > m$, we get that

$$k < 2 \times 10^{15} m\log(mk) < 4 \times 10^{15} m\log k,$$

therefore

$$\frac{k}{\log k} < 4 \times 10^{15} m. \tag{11}$$

Since the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that the inequality

$$\frac{x}{\log x} < A \qquad \text{yields} \qquad x < 2A\log A, \qquad \text{whenever} \quad A \geq 3.$$

Indeed, for if not, then we would have that $x > 2A\log A > e$, therefore

$$\frac{x}{\log x} > \frac{2A\log A}{\log(2A\log A)} > A,$$

where the last inequality follows because $2\log A < A$ holds for all $A \ge 3$. This is a contradiction. Hence, with these observations, we get that inequality (11) implies that

$$\begin{aligned} k &< 8\times 10^{15} m \log(4\times 10^{15} m) \\ &= 8\times 10^{15} m(\log(4\times 10^{15}) + \log m) \\ &< 8\times 10^{15} m(37+\log m). \end{aligned}$$

Since $\log m \ge \log 9 > 2$ and since $x+y < xy$ holds whenever $x \ge 2$ and $y \ge 2$, it follows that

$$k < 8\times 10^{15} \times 37 m \log m < 3\times 10^{17} m\log m. \tag{12}$$

This all was if $k > m$. The same inequality (12) holds obviously if $k \le m$ as well. Let us record this calculation for future use.

Lemma 1. *If $(k,m,n) \neq (2,3,5)$ is a nontrivial solution of equation (1) with $k \ge 2$, then $m \ge 9$ and the inequality*

$$\left|\alpha^n 5^{-1/2} F_m^{-k} - 1\right| < \frac{1}{2^{k-2}} \tag{13}$$

holds. Furthermore, the above inequality implies that $k < 3\times 10^{17} m\log m$ for all $m \ge 9$.

We next apply lattice basis reduction to the inequality (13) together with the condition that $m \le 130$. The result is an improved upper bound $k \le 13$. This works as follows. Let

$$\lambda = n\log\alpha - \log\sqrt{5} - k\log F_m.$$

Assuming for the moment that $k \ge 10$ we have by (13)

$$|\lambda| < 1.0025|e^\lambda - 1| < \frac{4.01}{2^k}, \tag{14}$$

and further we know for each $m = 9, 10, \ldots, 130$ the upper bounds

$$k < 3\times 10^{17} m\log m \qquad \text{and} \qquad n \le mk+1.$$

We take a number C slightly larger than the square of the upper bounds for k and n. Let

$$M := \begin{pmatrix} 1 & 0 \\ [C\log\alpha] & [C\log F_m] \end{pmatrix}, \qquad \text{and} \qquad y = \begin{pmatrix} 0 \\ [C(\log 5)/2] \end{pmatrix},$$

where $[\cdot]$ denotes rounding to the nearest integer. The columns of M span a lattice, and we apply lattice basis reduction (essentially the Euclidean algorithm, or, if you like, the 2-dimensional LLL) to it, in order to efficiently compute the lattice point closest to the point y. From this we find the distance d from y to the lattice. As $M\begin{pmatrix} n \\ -k\end{pmatrix}$ is a lattice point, this gives us that

$$\left\| M\begin{pmatrix} n \\ -k\end{pmatrix} - y\right\| \ge d.$$

Notice that

$$M\begin{pmatrix} n \\ -k \end{pmatrix} - y = \begin{pmatrix} n \\ \lambda^* \end{pmatrix}$$

where

$$\lambda^* = [C\log\alpha]n - [C\log F_m]k - [C(\log 5)/2] \quad \text{satisfies} \quad |\lambda^* - C\lambda| \le \frac{1}{2}(n+k+1).$$

Combining all the above inequalities we find that

$$|\lambda| > \frac{1}{C}\left(\sqrt{d^2 - (mk+1)^2} - ((m+1)k/2+1)\right), \tag{15}$$

which is useful only if d happens to be large enough. Then the upper and lower bounds (14), (15) for λ can be compared to yield a reduced upper bound for k. The reason for choosing C as roughly the square of the initial upper bounds is that the distance from a random point to a random 2-dimensional lattice can be expected to be roughly the square root of the lattice determinant, and (15) becomes useful when this distance is of the size of the initial upper bounds.

With $C := 10^{45}$ we computed for each m the distance d which varied from 7.06496×10^{21} to 2.13623×10^{23}. In all cases, we found $k \le 79$.

With this new upper bound for k we repeated the process. With $C := 10^8$ we again computed for each m the distance d which varied from 2642.56 to 50101.6. In all cases, we found $k \le 17$.

And we did it a third time, this time with $C := 10^6$, leading to $k \le 13$ (and even $k \le 10$ or 9 in most of the cases). Further reduction did not yield improvements. For more details on the lattice basis reduction algorithm and some worked out applications similar to the current one, see [**11**].

The factorizations of all Fibonacci numbers F_m with $m \le 130$ are known, so then we checked by brute force that $\sigma_k(F_m)$ is not a Fibonacci number for any $k \in \{2, \ldots, 13\}$ and $m \in \{9, \ldots, 130\}$. So, from now on we may assume that $m > 130$.

5. Another Linear Form in Logarithms

Let us look at the element

$$x = \frac{k}{\alpha^{2m}}.$$

Since $k < 3 \times 10^{17} m \log m$ and $m > 130$, it follows that

$$x < \frac{3 \times 10^{17} m \log m}{\alpha^{2m}} < \frac{1}{\alpha^m}, \tag{16}$$

where the last inequality holds for all $m \ge 97$. In particular, $x < 2 \times 10^{-21}$. We now write

$$F_m^k = \frac{\alpha^{mk}}{5^{k/2}}\left(1 - \frac{(-1)^m}{\alpha^{2m}}\right)^k.$$

If m is odd, then

$$\left(1-\frac{(-1)^m}{\alpha^{2m}}\right)^k=\left(1+\frac{1}{\alpha^{2m}}\right)^k<\exp(x)<1+2x$$

because $x<2\times 10^{-21}$ is very small. If m is even, then

$$\left(1-\frac{(-1)^m}{\alpha^{2m}}\right)^k=\exp\left(\log\left(1-\frac{1}{\alpha^{2m}}\right)k\right)>\exp(-2x)>1-2x,$$

again because x is very small. Thus, we have that

$$|F_m^k-\alpha^{mk}5^{-k/2}|<2x\alpha^{mk}5^{-k/2}.$$

The same argument together with the fact that x is small shows that

$$\frac{F_m^k}{\alpha^{mk}5^{-k/2}}\in[0.9,1.1].$$

Hence, returning to (7), we have

$$\left|\frac{\alpha^n}{5^{1/2}}-\frac{\alpha^{mk}}{5^{k/2}}\right|<2x\left(\frac{\alpha^{mk}}{5^{k/2}}\right)+\frac{1}{\sqrt{5}\alpha^n}+\frac{3F_m^k}{2^k},$$

and dividing the last relation across by $\alpha^{mk}5^{-k/2}$, we get

$$|\alpha^{n-mk}5^{(k-1)/2}-1|<\frac{2k}{\alpha^{2m}}+\frac{1.1}{\sqrt{5}\alpha^nF_m^k}+\frac{3\times 1.1}{2^k}. \tag{17}$$

The first term in the right hand side above is $<10^{-20}$ and the second one is even smaller since $n>m$, while the last term is $<3.3/4$. Thus, the right hand side is $<7/8$. Let

$$\Lambda=(mk-n)\log\alpha-(k-1)\log\sqrt{5}.$$

Thus $|e^\Lambda-1|<7/8$. If $\Lambda>0$, we get $|\Lambda|<|e^\Lambda-1|$. If $\Lambda<0$, we then get that

$$1-e^\Lambda<\frac{7}{8}.$$

In particular, $e^{|\Lambda|}<8$, and now $|\Lambda|<|e^{|\Lambda|}-1|=e^{|\Lambda|}|e^\Lambda-1|<8|e^\Lambda-1|$. To summarize, we have that

$$|\Lambda|<\frac{16k}{\alpha^{2m}}+\frac{8.8}{\sqrt{5}\alpha^nF_m^k}+\frac{3.3}{2^{k-3}}. \tag{18}$$

Using also inequality (16), we get that

$$|\Lambda|<\frac{16}{\alpha^m}+\frac{8.8}{\sqrt{5}\alpha^nF_m^k}+\frac{3.3}{2^{k-3}}. \tag{19}$$

The second term on the right hand side of the above inequality is smaller than the first term since $n\geq(m-2)k\geq 2m-4$ (see (9)), and $m>130$, so

$$\frac{8.8}{\sqrt{5}\alpha^nF_m^k}<\frac{8.8\alpha^4}{\sqrt{5}\alpha^{2m}F_9^2}<\frac{16}{\alpha^m}.$$

In particular, we record from (19), that both inequalities

$$|\Lambda|<10^{-19}+\frac{3.3}{2^{k-3}}. \tag{20}$$

and

$$|\Lambda| < \frac{32}{\alpha^m} + \frac{27}{2^k} < \frac{60}{\alpha^\lambda} \tag{21}$$

hold, where $\lambda = \min\{m, k\}$.

To find a lower bound on $|\Lambda|$, we use a Theorem of Laurent, Mignotte and Nesterenko (see Corollary 2 in [**5**]). That result asserts that if

$$\Lambda = b_1 \log \eta_1 - b_2 \log \eta_2$$

is nonzero, where $b_1,\ b_2$ are positive integers, and $\eta_1,\ \eta_2$ are algebraic numbers which are real, positive and multiplicatively independent, then

$$\log |\Lambda| \geq -24.34 D^2 \left(\max\left\{ \log b' + 0.14, \frac{21}{D}, \frac{1}{2} \right\} \right)^2 A_1 A_2 \tag{22}$$

where $D,\ A_1,\ A_2$ have the same meaning as in Theorem 2 (with $\alpha_1\ \alpha_2$ replaced by $\eta_1,\ \eta_2$, respectively), and

$$b' = \frac{b_1}{A_2} + \frac{b_2}{A_1}.$$

Unlike in Theorem 2, the result from [**5**] requires that

$$A_j \geq \max\{Dh(\eta_j), |\log \eta_j|, 1\} \qquad \text{for} \quad j = 1,\ 2.$$

For our application, we take $\eta_1 = \alpha,\ \eta_2 = 5^{1/2}$, so we can take $D = 2$, $A_1 = 1$ and $A_2 = \log 5$. We take $b_1 = mk - n$ and $b_2 = k - 1$. If $b_1 \leq 0$, it then follows directly that

$$(k-1)\log 5^{1/2} \leq |\Lambda| < 10^{-19} + \frac{3.3}{2^{k-3}},$$

which leads to $k \leq 3$. Otherwise, that is if $b_1 > 0$, then we can apply inequality (22) and get that

$$\log |\Lambda| > -24.34 \times 4 \left(\max\{\log b' + 0.14, 10.5\} \right)^2 \log 5.$$

We need an upper bound on b'. Observe that for $k \geq 100$, we have

$$b_1 \log \alpha < (k-1)\log 5^{1/2} + 10^{-19} + \frac{3.3}{2^{k-3}} < (k-1)\log 5^{1/2} + 0.0001,$$

so

$$b_1 < (k-1)\left(\frac{\log 5}{2\log \alpha}\right) + \frac{0.0001}{\log \alpha} < 1.7(k-1) + 0.01 < 1.7k.$$

Therefore

$$b' = \frac{b_1}{\log 5} + b_2 < \left(\frac{1.7}{\log 5}\right) k + k - 1 < 2.1k.$$

Observe that $\log b' + 0.14 < \log(2.1 \times e^{0.14} k) < \log(3k)$. Hence, we get that

$$\log |\Lambda| > -157 \left(\max\{\log(3k), 10.5\} \right)^2,$$

which combined with inequality (21) gives

$$\lambda \log \alpha - \log(60) < 157 \left(\max\{\log(3k), 10.5\} \right)^2. \tag{23}$$

Assume first that $\lambda = k$. When the maximum on the right hand side of (23) is 10.5, we get $k < 4 \times 10^4$, while when the maximum is $\log(3k)$, we get $k < 5 \times 10^4$. Hence, at any rate we get that $k < 5 \times 10^4$ when $k \leq m$. When $\lambda = m$, we get that

$m \le k$. When the maximum on the right hand side of (23) is 10.5, we get that $m \le 4 \times 10^4$, and by Lemma 1 we get that

$$k \le 3 \times 10^{17} \times 4 \times 10^4 \log(4 \times 10^4) < 2 \times 10^{23}.$$

When the maximum on the right hand side above is $\log(3k)$, we get that

$$m < \frac{1}{\log \alpha}\left(157(\log(3k))^2 + \log 60\right) < \frac{162}{\log \alpha}(\log(3k))^2 < 340(\log(3k))^2,$$

(because $\log(3k) \ge \log 6 > 1$ and $\log 60 < 5$), which together with Lemma 1 gives

$$k < 3 \times 10^{17} \times 340(\log(3k))^2 \log(340(\log(3k))^2),$$

giving $k < 5 \times 10^{24}$. Of course, if $k < 100$, then we have an even better inequality. So, the conclusion is that the inequality $k < 5 \times 10^{24}$ holds always.

We record this for future reference.

Lemma 2. *Assume that $m > 130$. Then*

$$|(mk - n)\log \alpha - (k-1)\log 5^{1/2}| < \frac{32k}{\alpha^{2m}} + \frac{3.3}{2^{k-3}}. \tag{24}$$

Furthermore, $k < 5 \times 10^{24}$.

Let us now get some improved bounds on k. Let $\Theta = (\log 5^{1/2})/\log \alpha$. Dividing (24) across by $(k-1)\log \alpha$, we get

$$\left|\frac{mk-n}{k-1} - \Theta\right| < \frac{32k}{(k-1)(\log \alpha)\alpha^{2m}} + \frac{3.3}{(k-1)(\log \alpha)2^{k-3}}. \tag{25}$$

Assume first that $k \ge 15$. Then

$$\frac{32k}{(\log \alpha)(k-1)\alpha^{2m}} < \left(\frac{32}{\log \alpha}\right)\left(\frac{15}{14}\right)\left(\frac{1}{\alpha^{260}}\right) < \frac{1}{1219(k-1)^2}, \quad (k < 5 \times 10^{24}).$$

Furthermore,

$$\frac{3.3}{(\log \alpha)(k-1)2^{k-3}} < \frac{1}{42.6(k-1)^2} \quad (k \ge 15).$$

Putting these together we get that estimate (25) implies

$$\left|\frac{mk-n}{k-1} - \Theta\right| < \frac{1}{1219(k-1)^2} + \frac{1}{42.6(k-1)^2} < \frac{1}{41(k-1)^2}. \tag{26}$$

Therefore by a classical criterion of Lagrange, $(mk-n)/(k-1)$ is a convergent of Θ for $k \ge 15$. Let the continued fraction of Θ be $[a_0, a_1, \ldots]$ with convergents $p_0/q_0,\ p_1/q_1, \ldots$. We have $q_{46} = 9.44778\ldots \times 10^{23}$ and $q_{47} = 6.28253\ldots \times 10^{24} > k$. Furthermore, $\max\{a_i : i = 0, \ldots, 47\} = 29$. Now $mk - n = \lambda p_i,\ k - 1 = \lambda q_i$ for some $i \in \{0, \ldots, 47\}$ and some natural number λ. We have

$$\left|\frac{mk-n}{k-1} - \Theta\right| = \left|\frac{p_i}{q_i} - \Theta\right| > \frac{1}{(a_{i+1}+2)q_i^2} = \frac{\lambda^2}{(a_{i+1}+2)(k-1)^2} \ge \frac{1}{31(k-1)^2},$$

contradicting inequality (26).

Thus, we have deduced so far that $k \le 14$. We return to (17), except that instead of $3/2^k$, we work directly with $\zeta(k) - 1$ (see (6)). We get

$$|\alpha^{n-mk}5^{(k-1)/2} - 1| < \frac{2k}{\alpha^{2m}} + \frac{1.1}{\sqrt{5}\alpha^n F_m^k} + 1.1(\zeta(k) - 1). \qquad (k \le 14). \qquad (27)$$

A case by case analysis shows that the only instances that survive are the following:

$$\begin{aligned} k &= 2, & n &= 2m-1,\ 2m-2,\ 2m-3,\ 2m-4, \\ k &= 3, & n &= 3m-3, \\ k &= 4, & n &= 4m-5. \end{aligned} \qquad (28)$$

We shall deal with these in the next section.

6. The Cases of the Small k

We start by treating the case when $6 \nmid m$. Then either F_n is odd, or $2 \| F_n$. Since $\sigma_k(F_m)$ is either odd or is divisible by 2 but not by 4, it follows easily that $F_m = \square,\ 2\square,\ p\square,\ 2p\square$, where here by $\square$ we mean a perfect square of an integer. It is well-known that if $F_m = \square$ or $2\square$, then $m \in \{1, 2, 3, 6, 12\}$ (see, for example, [**2**] for a more general result), contradicting the fact that $m > 130$. If m is such that $F_m = 2p\square$, then $m \le 36$ (see the beginning of Section 3 in [**1**]), again contradicting the fact that $m > 130$. It remains to treat the case $F_m = p\square$. By a result from [**10**], it follows that either $m \in \{4, 25\}$, which is not our case, or $m = q$ is a prime. Suppose that we are in this last case and write

$$F_q = \prod_{i=1}^{t} Q_i^{\gamma_i}$$

with primes $Q_1 < Q_2 < \cdots < Q_t$ and positive exponents $\gamma_1, \ldots, \gamma_t$. It is well-known that $Q_i \equiv \pm 1 \pmod{q}$, therefore $Q_1 \ge 2q - 1$. Furthermore,

$$(2q-1)^t \le \prod_{i=1}^{t} Q_i^{\gamma_i} = F_q < \alpha^q,$$

so $t < q \log \alpha / \log(2q - 1)$. Thus,

$$\begin{aligned} \frac{F_n}{F_q^k} &= \frac{\sigma_k(F_q)}{F_q^k} = \prod_{i=1}^{t} \left(\sum_{j=0}^{\gamma_i} \frac{1}{Q_i^{kj}} \right) \\ &< \prod_{i=1}^{t} \left(1 - \frac{1}{Q_i^k}\right)^{-1} = \prod_{i=1}^{t} \left(1 + \frac{1}{Q_i^k - 1}\right) < \left(1 + \frac{1}{(2q-1)^k - 1)}\right)^t \\ &< \exp\left(\frac{t}{(2q-1)^k - 1)}\right) < \exp\left(\frac{q \log \alpha}{((2q-1)^2 - 1)\log(2q-1)}\right) \\ &< 1 + \frac{2q \log \alpha}{((2q-1)^2 - 1)\log(2q-1)} < 1 + \frac{0.0005}{1.1}, \end{aligned}$$

where in the last inequalities above we used the fact that the inequality $e^x < 1 + 2x$ holds for all $x < 1/2$, as well as the fact that $q \ge 131$. This argument shows that the term $1.1 \times 3/2^k$ (or $1.1(\zeta(k)-1)$), can be replaced by 0.0005 in (27). Hence, we get the better inequality

$$|\alpha^{n-mk}5^{(k-1)/2}-1| < \frac{2k}{\alpha^{2m}} + \frac{1.1}{\sqrt{5}\alpha^n F_m^k} + 0.0005 < 0.0006, \qquad (k \le 4). \tag{29}$$

None of the possibilities listed in (28) passes this new test.

Next, we treat the case when $6 \mid n$. This eliminates immediately three of the cases in (28), namely then ones where $(k,n) = (2, 2m-1),\ (2, 2m-3),\ (4, 4m-5)$ for which n is odd.

Let us next eliminate the instance when $(k,n) = (3, 3m-3)$. In this case, n is even, so m is odd. If $3 \nmid m$, it follows that F_m is coprime to both 2 and 3. If $3 \mid m$, then $6 \nmid m$, so $2 \| F_m$. Thus, instead of $\zeta(3)$ in (27), we can work with

$$\eta_1 = \left(1 + \frac{1}{2^3}\right) \prod_{p \ge 5} \left(1 - \frac{1}{p^3}\right)^{-1}.$$

However, this case no longer passes the analog of inequality (27) with $\zeta(3)$ replaced by η_1.

Let us next deal with the case $(k,n) = (2, 2m-4)$. Since n is a multiple of 3, it follows that m isn't, therefore F_m is odd. Hence, instead of $\zeta(2)$ in (27), we can work with

$$\eta_2 = \prod_{p \ge 3} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

However, this case no longer passes the analog of inequality (27) with $\zeta(2)$ replaced by η_2.

Let us next deal with the last case $(k,n) = (2, 2m-2)$. Again since n is a multiple of 3, it follows that m is not a multiple of 3, therefore F_m is odd. Suppose first that 3 does not divide F_m. Then F_m is coprime to 6. Hence, we may replace in (27) the element $\zeta(2)$ by

$$\eta_3 = \prod_{p \ge 5} \left(1 - \frac{1}{p^2}\right)^{-1},$$

but then the analog inequality (27) is no longer satisfied. Assume next that $3 \mid F_m$, so $4 \mid m$. This shows that $2 \| n$, therefore $8 \| F_n = \sigma_k(F_m)$. This shows easily that $F_m = u\Box$, where u is odd, square-free, and has at most three distinct prime factors. Writing $m = 4m_1$, we get $F_{4m_1} = u\Box$, so $F_{m_1}L_{m_1}L_{2m_1} = u\Box$. Since m_1 is coprime to 3, it follows that $F_{m_1},\ L_{m_1}$ and L_{2m_1} are mutually coprime. If one of them is a square, then we get again that $m_1 \le 12$, so $m \le 48$, which is a contradiction. So the only chance is that $F_{m_1} = p_1\Box,\ L_{m_1} = p_2\Box$, and $L_{2m_1} = p_3\Box$ for some primes $p_1,\ p_2,\ p_3$. By the result from [**10**], either $m_1 \le 25$ (so, $m \le 100$, which is not allowed), or m_1 is prime. Clearly, $m_1 = m/4 \ge 33$. It then follows that $3 \| L_{2m_1}$, therefore $L_{2m_1} = 3\Box$. However, this is impossible with such a large m_1 by Theorem 2 in [**3**], for example.

This finishes the proof.

References

[1] K. A. Broughan, M. González, R. Lewis, F. Luca, V. J. Mejía Huguet and A. Togbé, *There are no multiply perfect Fibonacci numbers*, INTEGERS, to appear.

[2] Y. Bugeaud, M. Mignotte and S. Siksek, *Sur les nombres de Fibonacci de la forme $q^k y^p$*, C. R. Math. Acad. Sci. Paris, **339** (2004), 327–330.

[3] Y. Bugeaud, F. Luca, M. Mignotte and S. Siksek, *Almost powers in the Lucas sequence*, J. Théor. Nombres Bordeaux, **20** (2008), 555–600.

[4] S. Konyagin, F. Luca and P. Stănică, *Sums of divisors of Fibonacci numbers*, Uniform Distribution Theory, **4** (2009), 1–8.

[5] M. Laurent, M. Mignotte and Yu. Nesterenko, *Formes lineaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory, **55** (1995), 285–321.

[6] F. Luca, *Perfect Fibonacci and Lucas numbers*, Rend. Circ. Mat. Palermo **49** (2) (2000), 313–318.

[7] F. Luca, *Fibonacci numbers with the Lehmer property*, Bull. Polish Acad. Sci. Math., **55** (2007), 7–15.

[8] F. Luca and F. Nicolae, $\phi(F_n) = F_m$, INTEGERS, **9** (2009), 375–400.

[9] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II.* (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **64** no. 6 (2000), 125–180; translation in Izv. Math. **64** no. 6 (2000), 1217–1269.

[10] N. Robbins, *n Fibonacci numbers of the form px^2, where p is prime*, Fibonacci Quart.**21** (1983), 266–271.

[11] B. M. M. de Weger, *Algorithms for Diophantine Equations*, CWI Tract **65**, CWI Amsterdam, 1989.

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